Internat. J. Functional Analysis, Operator Theory and Applications

Published Online: June 2015
http://dx.doi.org/10.17654/IJFAOTAFeb2015_057_080
Volume 7, Number 1, 2015, Pages 57-80 **ISSN: 0975-2919**

UNIQUE COMMON FIXED POINT THEOREMS FOR SIX MAPPINGS UNDER CONTRACTIVE CONDITIONS OF INTEGRAL TYPE IN *G*-METRIC SPACES

R. A. Rashwan and H. A. Hammad

Department of Mathematics
Faculty of Science
Assuit University
Assuit 71516, Egypt
e-mail: rr_rashwan54@yahoo.com

Department of Mathematics
Faculty of Science
Sohag University
Sohag 82524, Egypt
e-mail: h_elmagd89@yahoo.com

Abstract

In this paper, we prove common fixed point theorems for six weakly compatible mappings satisfying the contractive conditions in the setting of generalized metric spaces satisfying an integral type and the common (E.A) property. Also, we redefine the concept of Wilson [1] and common (E.A) property in *G*-metric spaces. Our results generalize some well-known results in the literature.

Received: October 22, 2014; Revised: February 18, 2015; Accepted: March 28, 2015
2010 Mathematics Subject Classification: 47H10.
Keywords and phrases: metric spaces, generalized metric spaces, fixed point results, weakly compatible mappings.
Communicated by K. K. Azad

1. Introduction

The fixed point theorems in metric spaces are playing a major role to construct methods in mathematics to solve problems in applied mathematics and science. So the attraction of metric spaces to a large number of mathematicians is understandable. Some generalizations of the notation of a metric space have been proposed by some authors. In 1986, Jungck [2] proved a common fixed point theorem for commuting maps generalizing the Banach's fixed point theorem. This result was further generalized and extended in various ways by many others. Sessa [24] defined weak commutativity, also Jungck [2] introduced more generalized commutativity, so-called compatibility, which is more general than that of weak commutativity. Mustafa and Sims [3] generalized the concept of a metric space. Based on the notion of generalized metric spaces, Mustafa et al. [4-8] obtained some fixed point theorems for mappings satisfying different contractive conditions. For other results, we refer to [9-22]. Abbas and Rhoades [11] obtained some common fixed point theorems for non-commuting maps satisfying different contractive conditions in the setting of generalized metric space. Recently, Kaewcharoen [18] proved the existence and uniqueness of common fixed point theorems of a pair of weakly compatible mappings satisfying Φ-maps in generalized metric spaces, see also a recent paper of Shatanawi [22]. Commuting, weakly commuting, compatible mappings have been frequently used to prove existence theorems in common fixed point theory. Recall that, Jungck and Rhoades [23] defined S and T to be weakly compatible if they commute at there coincidence points i.e., if $Su = Tu$ for some $u \in X$, then $STu = TSu$.

Many examples in the literature illustrate that commuting implies weakly commuting implies compatible implies weakly compatible maps, but the converse need not be true (see [2] and [24]).

The following two axioms appeared in Wilson [1] for a symmetric space (X, d).

(W.3) Given $\{x_n\}$, x and y in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0$ imply $x = y$.

(W.4) Given $\{x_n\}$, $\{y_n\}$ and x in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$ imply that

$$\lim_{n\to\infty} d(y_n, x) = 0.$$

Aamri and Moutawakil [29] defined the property (E.A) and established some common fixed point theorems under strict contractive conditions on a metric space for mappings satisfying the property (E.A). Again recall that the pair (S, T) satisfies the property (E.A) if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \text{ for some } t \in X.$$

Most recently, Liu et al. [30] defined a common (E.A) property for a pair of mappings as follows:

Definition 1.1. Let $A, B, S, T : X \to X$. The pairs (A, S) and (B, T) satisfy the common (E.A) property if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t \in X.$$

Remark 1.1. If $B = A$ and $S = T$ in the above, then we obtain the definition of the property (E.A).

2. Basic Concepts

In this section, we present the necessary definitions and theorems in G-metric spaces.

Definition 2.1 [3]**.** Let X be a non-empty set, $G : X \times X \times X \to [0, +\infty)$ be a function satisfying the following axioms:

(G1) $G(x, y, z) = 0$ if $x = y = z$;

(G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;

(G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$;

(G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables);

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a *generalized metric*, or, more specially, a *G-metric* on *X*, and the pair (X, G) is called a *G-metric space*.

If condition (G6) also satisfied, then (X, G) is called *symmetric G-metric space*.

(G6) $G(x, y, y) \leq G(x, x, y)$ for all x, y in X.

Definition 2.2 [3]**.** Let (X, G) be a G-metric space, and let (x_n) be a sequence of points of X. We say that the sequence (x_n) is *G-convergent* to $x \in X$ if $\lim_{n, m \to +\infty} G(x, x_n, x_m) = 0$, that is, for any $\varepsilon > 0$, there exists a positive integer N such that $G(x, x_n, x_m) < \varepsilon$ for all $n, m \geq N$. We call x the *limit of the sequence* and write $x_n \to x$ or $\lim_{n \to +\infty} x_n = x$.

Proposition 2.1 [3]**.** *Let (X, G) be a G-metric space. Then the following are equivalent*:

(1) (x_n) *is G-convergent to x*;

(2) $G(x_n, x_n, x) \to 0$ *as* $n \to +\infty$;

(3) $G(x_n, x, x) \to 0$ *as* $n \to +\infty$;

(4) $G(x_n, x_m, x) \to 0$ *as* $n, m \to +\infty$.

Definition 2.3 [3]**.** Let (X, G) be a G-metric space. A sequence (x_n) is called a *G-Cauchy sequence* if, for any $\varepsilon > 0$, there is N such that $G(x_n, x_m, x_l) < \varepsilon$ for all $m, n \geq N$, that is, $G(x_n, x_m, x_l) \to 0$ as $n, m \to +\infty$.

Proposition 2.2 [3]**.** *Let* (X, G) *be a G-metric space. Then the following are equivalent*:

(1) *the sequence* (x_n) *is G-Cauchy*;

(2) *for any* $\varepsilon > 0$, *there exists N such that* $G(x_n, x_m, x_m) < \varepsilon$ *for all* $m, n \geq N$.

Definition 2.4 [3]**.** A *G*-metric space (X, G) is called *G-complete* if every *G*-Cauchy sequence is *G*-convergent in (X, G).

Every *G*-metric (X, G) defines a metric (X, d_G) by

$$d_G(x, y) = G(x, y, y) + G(y, x, x) \text{ for all } x, y \in X.$$

Definition 2.5 [3]**.** Let (X, G) and (X, G') be *G*-metric spaces, and $f : (X, G) \to (X, G')$ be a function. Then *f* is said to be *G-continuous* at a point $a \in X$ if and only if for every $\varepsilon > 0$, there is $\delta > 0$ such that $x, y \in X$, and $G(a, x, y) < \delta$ imply $G'(fa, fx, fy) < \varepsilon$. A function *f* is *G*-continuous at *X* if and only if it is *G*-continuous for all $a \in X$.

Definition 2.6 [25]**.** Self-mappings *f* and *g* of a *G*-metric space (X, G) are said to be *compatible* if $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$ and $\lim_{n\to\infty} G(gfx_n, fgx_n, fgx_n) = 0$, whenever $\{x_n\}$ is a sequence in *X* such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in X$.

In 2010, Manro et al. [26] introduced the concept of weakly commuting mappings, *R*-weakly commuting mappings in *G*-metric space as follows:

Definition 2.7 [26]**.** A pair of self-mappings *f*, *g* of a *G*-metric space is said to be *weakly commuting* if

$$G(fgx, gfx, gfx) \leq G(fx, gx, gx), \quad \forall x \in X.$$

Definition 2.8 [26]**.** A pair of self-mappings *f*, *g* of a *G*-metric space is said to be *R-weakly commuting*, if

$$G(fgx, gfx, gfx) \leq RG(fx, gx, gx), \quad \forall x \in X,$$

where $R > 0$.

Remark 2.1. If $R \leq 1$, then R-weakly commuting mappings are weakly commuting.

Remark 2.2. (i) Commuting mappings are weakly commuting mappings in G-metric spaces, but the reverse is not true (see [27]).

(ii) Weakly commuting mappings are R-weakly commuting mappings in G-metric spaces, but the reverse is not true (see [27]).

Definition 2.9. Let f and g be self-mappings of a set X. If $w = fx = gx$ for some x in X, then x is called a *coincidence point* of f and g.

Definition 2.10 [23]. $\varphi : [0, +\infty) \to [0, +\infty)$ is subadditive on each $[a, b] \subset [0, +\infty)$ if

$$\int_0^{a+b} \varphi(t)dt \leq \int_0^{a} \varphi(t)dt + \int_0^{b} \varphi(t)dt.$$

Shatanawi [20] proved the following theorem of G-metric spaces:

Theorem 2.1. *Let (X, G) be a G-metric space and $f : X \to X$ such that*

$$G(fx, fy, fz) \leq \phi(G(x, y, z))$$

for all $x, y, z \in X$, where ϕ is Φ-map. Then T has a unique fixed point (*say u*) *and T is G-continuous at u.*

Kaewcharoen [18] proved the following theorem in G-metric spaces:

Theorem 2.2. *Let (X, G) be a G-metric space and $f, g : X \to X$ such that*

$$G(fx, fy, fz) \leq \alpha G(gx, gy, gz)$$

for all $x, y, z \in X$, where $\alpha \in [0, 1)$. Assume that $f(X) \subset g(X)$ and

$g(X)$ is a complete subspace of X. Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

Aydi [28] gave the version of Kaewcharoen in integral type as follows:

Theorem 2.3. *Let (X, G) be a G-metric space and $f, g : X \to X$ such that*

$$\int_0^{G(fx, fy, fz)} u(t)dt \leq \alpha \int_0^{G(gx, gy, gz)} u(t)dt$$

for all $x, y \in X$, where $\alpha \in [0, 1)$ and $u : [0, +\infty) \to [0, +\infty)$ is a Lebesgue-integrable mapping which is summable, non-negative and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} u(t)dt > 0$.

Assume that $f(X) \subset g(X)$ and $g(X)$ is a complete subspace of X. Then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

In this paper, we redefine the concepts of Wilson [1] and common (E.A) property in *G*-metric spaces. Based on that, we prove common fixed point theorems for six weakly compatible mappings satisfying the contractive conditions of integral type in *G*-metric spaces.

3. Main Results

In the sequel, we need a function $\phi : R_+ \to R_+$ satisfying the condition $1 < \phi(t) < t$ for each $t > 0$.

Definition 3.1. Let (X, G) be a *G*-metric space, and let $\{x_n\}$ be a sequence of points of *X*. Then the two axioms of Wilson in *G*-metric space are as follows:

(W.3) Given $\{x_n\}$, x and y in X, $\lim_{n\to\infty} G(x_n, x_n, x) = 0$ and $\lim_{n\to\infty} G(x_n, x_n, y) = 0$ imply $x = y$.

(W.4) Given $\{x_n\}$, $\{y_n\}$ and x in X, $\lim_{n\to\infty} G(x_n, x_n, x) = 0$ and $\lim_{n\to\infty} G(x_n, x_n, y_n) = 0$ imply that

$$\lim_{n\to\infty} G(y_n, y_n, x) = 0.$$

Definition 3.2. Let (X, G) be a G-metric space and $A, B, S, T, g, h : X \to X$. The pairs (A, S), (B, T) and (g, h) satisfy the common (E.A) property if there exist three sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} gz_n$$
$$= \lim_{n\to\infty} hz_n = t \in X.$$

Now we present an example of the above definition as follows:

Example 3.1. Let $X = [0, 1]$ with $G(x, y, z) = |x - y| + |y - z| + |z - x|$ for all $x, y, z \in X$. Let $A, B, S, T, g, h : X \to X$ be self-maps on X defined by

$$A(x) = \begin{cases} 1 - \frac{5x}{8}, & \text{if } x \in [0, 1/2), \\ 1, & \text{if } x \in [0, 1/2), \end{cases}$$

$$S(x) = \begin{cases} 1 - 2x, & \text{if } x \in [0, 1/2), \\ 1, & \text{if } x \in [0, 1/2), \end{cases}$$

$$g(x) = \begin{cases} 1 - \frac{x}{3}, & \text{if } x \in [0, 1/2), \\ 1, & \text{if } x \in [0, 1/2), \end{cases}$$

$$h(x) = \begin{cases} 1 - \frac{3x}{4}, & \text{if } x \in [0, 1/2), \\ 1, & \text{if } x \in [0, 1/2), \end{cases}$$

$Tx = 1 - \frac{2x}{3}$ and $Bx = 1 - x$ for all $x \in X$.

Let $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ be the sequences defined by $x_n = \frac{1}{n}$, $y_n = \frac{1}{n+1}$ and $z_n = \frac{1}{n^2+1}$. Then

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} gz_n$$

$$= \lim_{n\to\infty} hz_n = 1 \in X.$$

Then the common (E.A) property is satisfied.

Now, we present our main result in the following theorems:

Theorem 3.1. *Let* (X, G) *be a G-metric space which satisfies* (W.3). *Let* A, B, S, T, h *and g be six self-mappings of X such that*

(i) $A(X) \subset T(X)$, $B(X) \subset S(X)$ *and* $h(X) \subset g(X)$,

(ii)

$$\int_0^{G(Ax, By, hz)} \varphi(t)dt \leq \Psi\left(\int_0^{aL(x, y, z)+(1-a)M(x, y, z)} \varphi(t)dt\right) \tag{3.1}$$

for all $x, y \in X$, $\Psi : R_+ \to R_+$ *such that* $0 < \Psi(t) < t$ *and* $\varphi : R_+ \to R_+$ *is a Lebesgue-integrable mapping which is summable*, *non-negative and such that*

$$\int_0^{\varepsilon} \varphi(t)dt > 0 \text{ for all } \varepsilon > 0, \tag{3.2}$$

where

$$L(x, y, z) = \max\{G(gx, Ty, Sz), G(gx, By, hz), G(By, Ty, Sz)\},$$

$$M(x, y, z) = [\max\{G^2(gx, Ty, Sz), G(gx, By, hz), G(By, Ty, Sz),$$
$$G(gx, Ty, Sz), G(gx, By, hz), G^2(By, Ty, Sz)\}]^{1/2},$$

and $0 \leq a \leq 1$. *Suppose that* (A, g), (B, T) *and* (h, S) *satisfy the common* $(E.A)$ *property and are weakly compatible. If one of the subspaces* AX, gX, BX, TX, SX *and hX of X is complete, then* A, g, B, T, S *and h have a unique common fixed point in X.*

Proof. Since (A, g), (B, T) and (h, S) satisfy the common $(\mathrm{E.A})$ property, there exist three sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that

$$\lim_{n\to\infty} G(Ax_n, z, z) = \lim_{n\to\infty} G(Sz_n, z, z) = \lim_{n\to\infty} G(By_n, z, z)$$

$$= \lim_{n\to\infty} G(Ty_n, z, z) = \lim_{n\to\infty} G(gx_n, z, z)$$

$$= \lim_{n\to\infty} G(hz_n, z, z) = 0 \tag{3.3}$$

for some $z \in X$.

Now, suppose that gX is a complete subspace of X. Then $z = gu$ for some $u \in X$.

Consequently, we have

$$\lim_{n\to\infty} G(Ax_n, gu, gu)$$

$$= \lim_{n\to\infty} G(Sz_n, gu, gu) = \lim_{n\to\infty} G(By_n, gu, gu) = \lim_{n\to\infty} G(Ty_n, gu, gu)$$

$$= \lim_{n\to\infty} G(gx_n, gu, gu) = \lim_{n\to\infty} G(hz_n, gu, gu) = 0. \tag{3.4}$$

If $Au \neq z$ and using (3.1), then we get

$$\int_0^{G(Au, By_n, hz_n)} \varphi(t)dt \leq \Psi\left(\int_0^{aL(u, y_n, z_n)+(1-a)M(u, y_n, z_n)} \varphi(t)dt\right)$$

$$< \int_0^{aL(u, y_n, z_n)+(1-a)M(u, y_n, z_n)} \varphi(t)dt, \tag{3.5}$$

where

$$L(u, y_n, z_n)$$
$$= \max\{G(gu, Ty_n, Sz_n), G(gu, By_n, hz_n), G(By_n, Ty_n, Sz_n)\},$$

$$M(u, y_n, z_n) = [\max\{G^2(gu, Ty_n, Sz_n), G(gu, By_n, hz_n), G(By_n, Ty_n, Sz_n),$$
$$G(gu, Ty_n, Sz_n), G(gu, By_n, hz_n), G^2(By_n, Ty_n, Sz_n)\}]^{1/2}.$$

Taking $n \to \infty$, we get $L(u, y_n, z_n) = 0$ and $M(x, y, z) = 0$, respectively. Now, (3.5) becomes

$$\lim_{n\to\infty} \int_0^{G(Au, By_n, gz_n)} \varphi(t)\,dt = 0,$$

and (3.2) implies that $\lim_{n\to\infty} G(Au, By_n, gz_n) = 0$. By (W.3), we have $z = Au = gu$.

Since $AX \subset TX$, there exists $v \in X$ such that $z = Au = Tv$.

If $Bv \neq z$, using (3.1), we have

$$\int_0^{G(z, Bv, gz_n)} \varphi(t)\,dt = \int_0^{G(Au, Bv, hz_n)} \varphi(t)\,dt$$
$$\leq \Psi\left(\int_0^{aL(u, v, z_n)+(1-a)M(u, v, z_n)} \varphi(t)\,dt\right), \tag{3.6}$$

where

$$L(u, v, z_n) = \max\{G(gu, Tv, Sz_n), G(gu, Bv, hz_n), G(Bv, Tv, Sz_n)\},$$

$$M(u, v, z_n) = [\max\{G^2(gu, Tv, Sz_n), G(gu, Bv, hz_n), G(Bv, Tv, Sz_n),$$
$$G(gu, Tv, Sz_n), G(gu, Bv, hz_n), G^2(Bv, Tv, Sz_n)\}]^{1/2},$$

from which we get $L(u, v, z_n) = G(z, Bv, z)$ and

$$M(u, v, z_n) = [G^2(z, Bv, z)]^{1/2},$$

respectively. Now (3.6) becomes

$$\int_0^{G(z, Bv, z)} \varphi(t)dt = \int_0^{G(Au, Bv, hz_n)} \varphi(t)dt$$

$$\leq \Psi\left(\int_0^{aG(z, Bv, z)+(1-a)G(z, Bv, z)} \varphi(t)dt\right)$$

$$= \Psi\left(\int_0^{G(z, Bv, z)} \varphi(t)dt\right) < \int_0^{G(z, Bv, z)} \varphi(t)dt$$

which is a contradiction. Hence,

$$\int_0^{G(z, Bv, z)} \varphi(t)dt = 0,$$

and (3.2) implies that $\lim_{n\to\infty} G(z, Bv, z) = 0,$ i.e., $z = Bv = Tv.$

Since (A, g) is weakly compatible, so $Agu = gAu$ whenever $Au = gu$ which implies

$$Az = gz. \tag{3.7}$$

Let us show that z is a common fixed point of A, g, B, T, S and h.

If $z \neq Az,$ again using (3.1), we get

$$\int_0^{G(Az, z, z)} \varphi(t)dt = \int_0^{G(Az, Bv, hz_n)} \varphi(t)dt$$

$$\leq \Psi\left(\int_0^{aL(z, v, z_n)+(1-a)M(z, v, z_n)} \varphi(t)dt\right), \tag{3.8}$$

where

$$L(z, v, z_n) = \max\{G(gz, Tv, Sz_n), G(gz, Bv, hz_n), G(Bv, Tv, Sz_n)\},$$

$$M(z, v, z_n) = [\max\{G^2(gz, Tv, Sz_n), G(gz, Bv, hz_n), G(Bv, Tv, Sz_n),$$

$$G(gz, Tv, Sz_n), G(gz, Bv, hz_n), G^2(Bv, Tv, Sz_n)\}]^{1/2},$$

from which we get $L(z, v, z_n) = G(Az, z, z)$ and

$$M(u, v, z_n) = [G^2(Az, z, z)]^{1/2},$$

respectively. Now (3.8) becomes

$$\int_0^{G(Az, z, z)} \varphi(t)dt = \int_0^{G(Az, Bv, hz_n)} \varphi(t)dt$$

$$\leq \Psi\left(\int_0^{aG(Az, z, z)+(1-a)G(Az, z, z)} \varphi(t)dt\right)$$

$$= \Psi\left(\int_0^{G(Az, z, z)} \varphi(t)dt\right) < \int_0^{G(Az, z, z)} \varphi(t)dt$$

which is a contradiction. Therefore,

$$\int_0^{G(Az, z, z)} \varphi(t)dt = 0,$$

and (3.2) implies that $G(Az, z, z) = 0$, i.e., $z = Az = gz$.

Similarly, the weak compatibility of B and T implies $BTv = TBv$, i.e., $Bz = Tz$. If $z \neq Bz$, by using (3.1) and (3.2), a similar calculation to the above yields $z = Bz = Tz$. Thus, z is a common fixed point of A, g, B, T, S and h.

When TX is assumed to be a complete subspace of X, then the proof is similar. On the other hand, the cases in which AX or BX and SX or hX is a complete subspace of X are similar to the cases in which TX or gX is complete, respectively, by (3.1).

For the uniqueness of the common fixed point z, let $w \neq z$ be another common fixed point of A, g, B, T, S and h. Then, using (3.1), we obtain

$$\int_0^{G(w, z, z)} \varphi(t)dt = \int_0^{G(Aw, Bz, hz)} \varphi(t)dt$$

$$\leq \Psi\left(\int_0^{aL(w, z, z)+(1-a)M(w, z, z)} \varphi(t)dt\right),$$

$$\Psi\left(\int_0^{aG(w,z,z)+(1-a)G(w,z,z)} \varphi(t)\,dt\right)$$

$$= \Psi\left(\int_0^{G(w,z,z)} \varphi(t)\,dt\right) < \int_0^{G(w,z,z)} \varphi(t)\,dt$$

which is a contradiction. Therefore, $\int_0^{G(w,z,z)} \varphi(t)\,dt = 0,$ and (3.2) implies that $z = w.$

For $\varphi(t) = 1$ in Theorem 3.1, we obtain the following corollary:

Corollary 3.1. *Let* (X, G) *be a G-metric space which satisfies* (W.3). *Let* A, B, S, T, h *and g be six self-mappings of X such that*

(i) $A(X) \subset T(X),\ B(X) \subset S(X)$ *and* $h(X) \subset g(X),$

(ii) $G(Ax, By, hz) \leq \Psi(aL(x, y, z) + (1-a)M(x, y, z))$

for all $x, y \in X,$ *where*

$$L(x, y, z) = \max\{G(gx, Ty, Sz), G(gx, By, hz), G(By, Ty, Sz)\},$$

$$M(x, y, z) = [\max\{G^2(gx, Ty, Sz), G(gx, By, hz), G(By, Ty, Sz),$$

$$G(gx, Ty, Sz), G(gx, By, hz), G^2(By, Ty, Sz)\}]^{1/2},$$

and $0 \leq a \leq 1.$ *Suppose that* $(A, g), (B, T)$ *and* (h, S) *satisfy the common* (*E.A*) *property and are weakly compatible*. *If one of the subspaces AX*, *gX*, *BX*, *TX*, *SX and hX of X is complete*, *then A*, *g*, *B*, *T*, *S and h have a unique common fixed point in X*.

If we take $A(X) = B(X) = h(X)$ and $T(X) = S(X) = g(X)$ in Theorem 3.1, then we get the following corollary:

Corollary 3.2. *Let* (X, G) *be a G-metric space which satisfies* (W.3). *Let A and B be self-mappings of X such that*

(i) $A(X) \subset T(X)$,

(ii)

$$\int_0^{G(Ax, Ay, Az)} \varphi(t)\,dt \leq \Psi\left(\int_0^{aL(x, y, z)+(1-a)M(x, y, z)} \varphi(t)\,dy\right)$$

for all $x, y \in X$, *where* $\varphi : R_+ \to R_+$ *is a Lebesgue-integrable mapping which is summable, non-negative and such that*

$$\int_0^{\varepsilon} \varphi(t)\,dt > 0 \text{ for all } \varepsilon > 0,$$

where

$$L(x, y, z) = \max\{G(Tx, Ty, Tz), G(Tx, Ay, Az), G(Ay, Ty, Tz)\},$$

$$M(x, y, z) = [\max(G^2(Tx, Ty, Tz), G(Tx, Ay, Az), G(Ay, Ty, Tz),$$

$$G(Tx, Ty, Tz), G(Tx, Ay, Az), G^2(Ay, Ty, Tz)\}]^{1/2},$$

and $0 \leq a \leq 1$. *Suppose that* (A, T) *satisfies the common* $(E.A)$ *property and is weakly compatible. If one of the subspaces AX or TX of X is complete, then A and T have a unique common fixed point in X.*

For $\varphi(t) = 1$ in Corollary 3.2, we obtain the following corollary:

Corollary 3.3. *Let* (X, G) *be a G-metric space which satisfies* (W.3). *Let A and B be self-mappings of X such that*

(i) $A(X) \subset T(X)$,

(ii) $G(Ax, Ay, Az) \leq \Psi(aL(x, y, z) + (1-a)M(x, y, z))$

for all $x, y \in X$, *where*

$$L(x, y, z) = \max\{G(Tx, Ty, Tz), G(Tx, Ay, Az), G(Ay, Ty, Tz)\},$$

$$M(x, y, z) = [\max(G^2(Tx, Ty, Tz), G(Tx, Ay, Az), G(Ay, Ty, Tz),$$

$$G(Tx, Ty, Tz), G(Tx, Ay, Az), G^2(Ay, Ty, Tz)\}]^{1/2},$$

and $0 \leq a \leq 1$. *Suppose that* (A, T) *satisfies the common* $(E.A)$ *property and is weakly compatible. If one of the subspaces AX or TX of X is complete, then A and T have a unique common fixed point in X.*

Remark 3.1. (i) If we take $a = 1$, $L(x, y, z) = G(Tx, Ty, Tz)$ and $\Psi(t) = \alpha(t)$ in Corollary 3.2, then we have (Theorem 2.2 in [26]).

(ii) If we take $a = 1$, $L(x, y, z) = G(Tx, Ty, Tz)$ and $\Psi(t) = \alpha(t)$ in Corollary 3.3, then we have (Theorem 2.1 in [28]).

Now, we give an example to support Corollary 3.1.

Example 3.2. Let $X = [0, \infty)$, $G(x, y, z) = |x - y| + |y - z| + |z - x|$ for all $x, y, z \in X$. Let $A, B, S, T, g, h : X \to X$ be defined by $Ax = x/8$, $Bx = x/16$, $Tx = x/2$, $Sx = x/4$, $hx = x/32$, $gx = x$. Clearly, we can get $A(X) \subset T(X)$, $B(X) \subset S(X)$ and $h(X) \subset g(X)$, also, (A, g), (B, T) and (h, S) satisfy the common (E.A) property as

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} gx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = \lim_{n\to\infty} Sz_n$$

$$= \lim_{n\to\infty} hz_n = 0 \in X,$$

if $\{x_n\}, \{y_n\}$ and $\{z_n\}$ are the sequences defined by $x_n = \frac{1}{n^2}$, $y_n = \frac{1}{n^3 + 1}$ and $z_n = \frac{1}{n^4 + 1}$, and are weakly compatible.

If we take $\Psi(t) = t/2$, $x = 0$, $y = 1$ and $z = 2$, then we have $\Psi(t) = t/2 < t$ for each $t > 0$, and

$$G(Ax, By, hz) = G(x/8, y/16, z/32) = \frac{1}{8}. \tag{3.9}$$

$L(x, y, z) = 1$ and $M(x, y, z) = 1$,

$$\phi(aL(x, y, z) + (1 - a)M(x, y, z)) = \phi(1) = \frac{1}{2}, \tag{3.10}$$

from (3.9) and (3.10), we get the inequality of Corollary 3.1 is satisfied, so that all the conditions of Corollary 3.1 are satisfied. Moreover, 0 is the unique common fixed point for all the mappings *A*, *B*, *S*, *T*, *g* and *h*.

Theorem 3.2. *Let* (X, G) *be a complete G-metric space and f, g, S, T, h and R be six self-mappings of X such that*

(i) $f(X) \subseteq g(X)$, $h(X) \subseteq S(X)$ *and* $R(X) \subseteq T(X)$,

(ii) *one of the six mappings is continuous*,

(iii) *the pairs* (f, g), (h, S) *and* (R, T) *are weakly compatible*,

(iv)

$$\int_0^{G(fx, hy, Rz)} \varphi(t)dt \leq \alpha \int_0^{G(fx, Sy, Tz)} \varphi(t)dt$$

$$+ \beta \int_0^{G(gx, hy, Tz)} \varphi(t)dt + \gamma \int_0^{G(gx, Sy, Rz)} \varphi(t)dt, \quad (3.11)$$

for every $x, y, z \in X$, $0 \leq \alpha + 3\beta + 3\gamma < 1$ *and* $\varphi : R_+ \to R_+$ *is a Lebesgue-integrable mapping which is summable and subadditive subset of* R_+, *non-negative and such that*

$$\int_0^{\varepsilon} \varphi(t)dt > 0 \text{ for all } \varepsilon > 0.$$

Then f, g, S, T, h and R have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in *X*. From (i), there exist $x_1, x_2, x_3 \in X$ such that $fx_0 = gx_1$, $hx_1 = Sx_2$, $Rx_2 = Tx_3$. By induction, there exist sequences $\{x_n\}$ and $\{y_n\}$ in *X* such that

$$y_{3n+1} = fx_{3n} = gx_{3n+1}, \quad y_{3n+2} = hx_{3n+1} = Sx_{3n+2}$$

$$\text{and } y_{3n+3} = Rx_{3n+2} = Tx_{3n+3}, \quad n = 0, 1, 2, ... \quad (3.12)$$

If $y_{3n+1} = y_{3n+2}$, then $hx = gx$, $x = x_{3n+1}$.

If $y_{3n+2} = y_{3n+3}$, then $Rx = sx$, $x = x_{3n+2}$.

If $y_{3n+1} = y_{3n}$, then $Tx = fx$, $x = x_{3n}$.

If $y_{n+1} \neq y_n$, then from (3.11) and using (3.12), we get

$$\int_0^{G(y_{3n}, y_{3n+1}, y_{3n+1})} \varphi(t)dt$$

$$= \int_0^{G(fx_{3n-1}, hx_{3n}, Rx_{3n})} \varphi(t)dt$$

$$\leq \alpha\int_0^{G(fx_{3n-1}, Sx_{3n}, Tx_{3n})} \varphi(t)dt + \beta\int_0^{G(gx_{3n-1}, hx_{3n}, Tx_{3n})} \varphi(t)dt$$

$$+ \gamma\int_0^{G(gx_{3n-1}, Sx_{3n}, Rx_{3n})} \varphi(t)dt$$

$$= \alpha\int_0^{G(y_{3n}, y_{3n}, y_{3n})} \varphi(t)dt + \beta\int_0^{G(y_{3n-1}, y_{3n+1}, y_{3n})} \varphi(t)dt$$

$$+ \gamma\int_0^{G(y_{3n-1}, y_{3n}, y_{3n+1})} \varphi(t)dt$$

$$= (\beta + \gamma)\int_0^{G(y_{3n-1}, y_{3n}, y_{3n+1})} \varphi(t)dt, \tag{3.13}$$

from the rectangular inequality, we have

$$G(y_{3n-1}, y_{3n}, y_{3n+1}) \leq G(y_{3n-1}, y_{3n}, y_{3n}) + G(y_{3n}, y_{3n}, y_{3n+1})$$

$$\leq G(y_{3n-1}, y_{3n}, y_{3n}) + 2G(y_{3n}, y_{3n+1}, y_{3n+1}), \tag{3.14}$$

from (3.13) and using (3.14), we obtain

$$\int_0^{G(y_{3n}, y_{3n+1}, y_{3n+1})} \varphi(t)dt \leq \frac{\beta + \gamma}{1 - 2\beta - 2\gamma}\int_0^{G(y_{3n-1}, y_{3n}, y_{3n})} \varphi(t)dt,$$

that is,

$$\int_0^{G(y_{3n}, y_{3n+1}, y_{3n+1})} \varphi(t)\,dt \le r \int_0^{G(y_{3n-1}, y_{3n}, y_{3n})} \varphi(t)\,dt,$$

where $r = \dfrac{\beta + \gamma}{1 - 2\beta - 2\gamma} < 1.$

By induction, one can write that

$$\int_0^{G(y_n, y_{n+1}, y_{n+1})} \varphi(t)\,dt \le r^n \int_0^{G(y_0, y_1, y_1)} \varphi(t)\,dt.$$

Therefore, for all $n, m \in N$, $n < m$, we get by rectangular inequality and properties of φ that

$$\int_0^{G(y_n, y_m, y_m)} \varphi(t)\,dt \le \int_0^{G(y_n, y_{n+1}, y_{n+1})} \varphi(t)\,dt + \int_0^{G(y_{n+1}, y_{n+1}, y_{n+2})} \varphi(t)\,dt$$

$$+ \cdots + \int_0^{G(y_{m-1}, y_m, y_m)} \varphi(t)\,dt$$

$$\le (r^n + r^{n+1} + \cdots + r^{m-1}) \int_0^{G(y_0, y_1, y_1)} \varphi(t)\,dt,$$

taking the limit as $n, m \to \infty$, we have

$$\lim_{n\to\infty} \int_0^{G(y_n, y_m, y_m)} \varphi(t)\,dt = 0 \text{ or } \lim_{n\to\infty} G(y_n, y_m, y_m) = 0, \text{ from (3.2).}$$

Thus, $\{y_n\}$ is a G-Cauchy sequence in X. So it is convergent. Let its limit be z, i.e.,

$$\lim_{n\to\infty} y_{3n+1} = \lim_{n\to\infty} fx_{3n} = \lim_{n\to\infty} gx_{3n+1} = z,$$

$$\lim_{n\to\infty} y_{3n+2} = \lim_{n\to\infty} hx_{3n+1} = \lim_{n\to\infty} Sx_{3n+2} = z,$$

$$\lim_{n\to\infty} y_{3n+3} = \lim_{n\to\infty} Rx_{3n+2} = \lim_{n\to\infty} Tx_{3n+3} = z$$

$$\Rightarrow \lim_{n\to\infty} fx_{3n} = \lim_{n\to\infty} gx_{3n+1} = \lim_{n\to\infty} hx_{3n+1}$$

$$= \lim_{n\to\infty} Sx_{3n+2} = \lim_{n\to\infty} Rx_{3n+2} = \lim_{n\to\infty} Tx_{3n+3} = z. \quad (3.15)$$

If g is continuous, then

$$\lim_{n\to\infty} ggx_{3n+1} = \lim_{n\to\infty} ghx_{3n+1} = gz \text{ (we denote } x_{3n} \to x_n).$$

Further, f and g are weakly compatible, therefore

$$\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$$

implies

$$\lim_{n\to\infty} fgx_n = gz. \quad (3.16)$$

If $z \neq gz$, then from (3.11), we get

$$\int_0^{G(fgx_{n-1}, hgx_n, Rx_n)} \varphi(t)dt \leq \alpha \int_0^{G(fgx_{n-1}, Sx_n, Tx_n)} \varphi(t)dt$$

$$+ \beta \int_0^{G(ggx_{n-1}, hx_n, Tx_n)} \varphi(t)dt$$

$$+ \gamma \int_0^{G(ggx_{n-1}, Sx_n, Rx_n)} \varphi(t)dt$$

as $n \to \infty$, and using (3.15), (3.16), we have

$$\int_0^{G(gz, z, z)} \varphi(t)dt \leq (\alpha + \beta + \gamma) \int_0^{G(gz, z, z)} \varphi(t)dt$$

which is a contradiction (since $\alpha + \beta + \gamma < 1$), therefore

$$\int_0^{G(gz, z, z)} \varphi(t)dt = 0,$$

from (3.2), we have $G(gz, z, z) = 0$ or $gz = z$.

Again, if $z \neq fz,$ then from (3.11), we get

$$\int_0^{G(fz, hx_n, Rx_n)} \varphi(t)dt \leq \alpha \int_0^{G(fz, Sx_n, Tx_n)} \varphi(t)dt$$

$$+ \beta \int_0^{G(gz, hx_n, Tx_n)} \varphi(t)dt + \gamma \int_0^{G(gz, Sx_n, Rx_n)} \varphi(t)dt,$$

taking the limit as $n \to \infty,$ and using (3.15), (3.16), we have

$$\int_0^{G(fz, z, z)} \varphi(t)dt \leq \alpha \int_0^{G(fz, z, z)} \varphi(t)dt$$

which is a contradiction, therefore $\int_0^{G(gz, z, z)} \varphi(t)dt = 0,$ from (3.2), we have $G(fz, z, z) = 0$ or $fz = z$

$$\Rightarrow gz = fz = z. \tag{3.17}$$

By the same way, if we take h or S is continuous and the pair (h, S) is weakly compatible, then

$$hv = Sv = v. \tag{3.18}$$

Also, if we take T or R is continuous and the pair (T, R) is weakly compatible, therefore

$$T\mu = R\mu = \mu. \tag{3.19}$$

If $z \neq v \neq \mu,$ then from (3.11), we get

$$\int_0^{G(z, v, \mu)} \varphi(t)dt = \int_0^{G(fz, hv, R\mu)} \varphi(t)dt \leq \alpha \int_0^{G(fz, Sv, T\mu)} \varphi(t)dt$$

$$+ \beta \int_0^{G(gz, hv, T\mu)} \varphi(t)dt + \gamma \int_0^{G(gz, Sv, R\mu)} \varphi(t)dt,$$

by using (3.18) and (3.19), we have

$$\int_0^{G(z, v, \mu)} \varphi(t)dt \leq (\alpha + \beta + \gamma) \int_0^{G(z, v, \mu)} \varphi(t)dt,$$

we have a contradiction again, then from (3.2), $G(z, v, \mu) = 0$ or $z = v = \mu$, i.e.,

$$hz = Sz = gz = fz = Tz = Rz = z.$$

Uniqueness follows easily from (3.11).

Therefore, f, g, S, T, h and R have a unique common fixed point in X.

References

[1] W. A. Wilson, On semi-metric spaces, Amer. J. Math. 53 (1931), 361-373.

[2] G. Jungck, Compatible mappings and common fixed points, Int. J. Math. Math. Sci. 9 (1986), 771-779.

[3] Z. Mustafa and B. Sims, A new approach to generalized metric spaces, J. Nonlinear Convex Anal. 7(2) (2006), 289-297.

[4] Z. Mustafa, A new structure for generalized metric spaces with applications to fixed point theory, Ph.D. Thesis, The University of Newcastle, Callaghan, Australia, 2005.

[5] Z. Mustafa and B. Sims, Some remarks concerning *D*-metric spaces, Proceedings Int. Conference Fixed Point Theory Appl., Yokohama, Japan, 2004, pp. 189-198.

[6] Z. Mustafa, H. Obiedat and F. Awawdeh, Some fixed point theorem for mapping on complete *G*-metric spaces, Fixed Point Theory Appl. 2008 (2008), 1-12.

[7] Z. Mustafa and B. Sims, Fixed point theorems for contractive mappings in complete *G*-metric spaces, Fixed Point Theory Appl. 2009 (2009), 1-10.

[8] Z. Mustafa, W. Shatanawi and M. Bataineh, Existence of fixed point results in *G*-metric spaces, Int. J. Math. Math. Sci. 2009 (2009), 1-10.

[9] M. Abbas, A. R. Khan and T. Nazir, Coupled common fixed point results in two generalized metric spaces, Appl. Math. Comput. 217 (2011), 6328-6336.

[10] M. Abbas, S. H. Khan and T. Nazir, Common fixed points of *R*-weakly commuting maps in generalized metric spaces, Fixed Point Theory Appl. 2011: 41 (2011), 1-11.

[11] M. Abbas and B. E. Rhoades, Common fixed point results for non-commuting mappings without continuity in generalized metric spaces, Appl. Math. Comput. 215 (2009), 262-269.

[12] H. Aydi, B. Damjanović, B. Samet and W. Shatanawi, Coupled fixed point theorems for nonlinear contractions in partially ordered *G*-metric spaces, Math. Comput. Modelling 54 (2011), 2443-2450.

[13] H. Aydi, W. Shatanawi and C. Vetro, On generalized weakly *G*-contraction mapping in *G*-metric spaces, Comput. Math. Appl. 62 (2011), 4222-4229.

[14] H. Aydi, W. Shatanawi and M. Postolache, Coupled fixed point results for (Ψ, ϕ)- weakly contractive mappings in ordered *G*-metric spaces, Comput. Math. Appl. 63 (2012), 298-309.

[15] H. Aydi, Common fixed point results for nonlinear contractions in *G*-metric spaces, Mat. Vesnik, in printing.

[16] B. S. Choudhury and P. Maity, Coupled fixed point results in generalized metric spaces, Math. Comput. Modelling 54 (2011), 73-79.

[17] B. C. Dhage, Generalized metric space and mapping with fixed point, Bull. Calcutta Math. Soc. 84 (1992), 329-336.

[18] A. Kaewcharoen, Common fixed point theorems for contractive mappings satisfying ϕ-maps in *G*-metric spaces, Banach J. Math. Anal. 6 (2012), 101-111.

[19] R. Saadati, S. M. Vaezpour, P. Vetro and B. E. Rhoades, Fixed point theorems in generalized partially ordered *G*-metric spaces, Math. Comput. Modelling 5 (2010), 797-801.

[20] W. Shatanawi, Fixed point theory for contractive mappings satisfying φ-maps in *G*-metric spaces, Fixed Point Theory Appl. 2010 (2010), 1-9.

[21] W. Shatanawi, Some fixed point theorems in ordered *G*-metric spaces and applications, Abstr. Appl. Anal. 2011 (2011), 1-11.

[22] W. Shatanawi, Common fixed point results for two self-maps in *G*-metric spaces, Mat. Vesnik, in printing.

[23] G. Jungck and B. E. Rhoades, Fixed points for set valued functions without continuity, Indian J. Pure. Appl. Math. 29(3) (1998), 227-238.

[24] S. Sessa, On a weak commutativity condition of mappings in fixed point considerations, Publ. Inst. Math. (Beograd) 32(46) (1982), 149-153.

[25] R. K. Vats, S. Kumar and V. Sihag, Some common fixed point theorems for compatible mappings of type *A* in complete *G*-metric space, Advances in Fuzzy Mathematics 6(1) (2011), 27-38.

[26] S. Manro, S. S. Bhatia and S. Kumar, Expansion mappings theorems in *G*-metric spaces, Int. J. Contemp. Math. Sci. 5(51) (2010), 2529-2535.

[27] Feng Gu and Hongqing Ye, Common Fixed Point Theorems of Altman Integral Type Mappings in *G*-Metric Spaces, Abstr. Appl. Anal. 2012 (2012), 1-13.

[28] H. Aydi, A common fixed point of integral type contraction in generalized metric spaces, J. Adv. Math. Stud. 5(1) (2012), 111-117.

[29] M. Aamri and D. El Moutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl. 270 (2002), 181-188.

[30] W. Liu, J. Wu and Z. Li, Common fixed points of single-valued and multi-valued maps, Int. J. Math. Math. Sci. 19 (2005), 3045-3055.